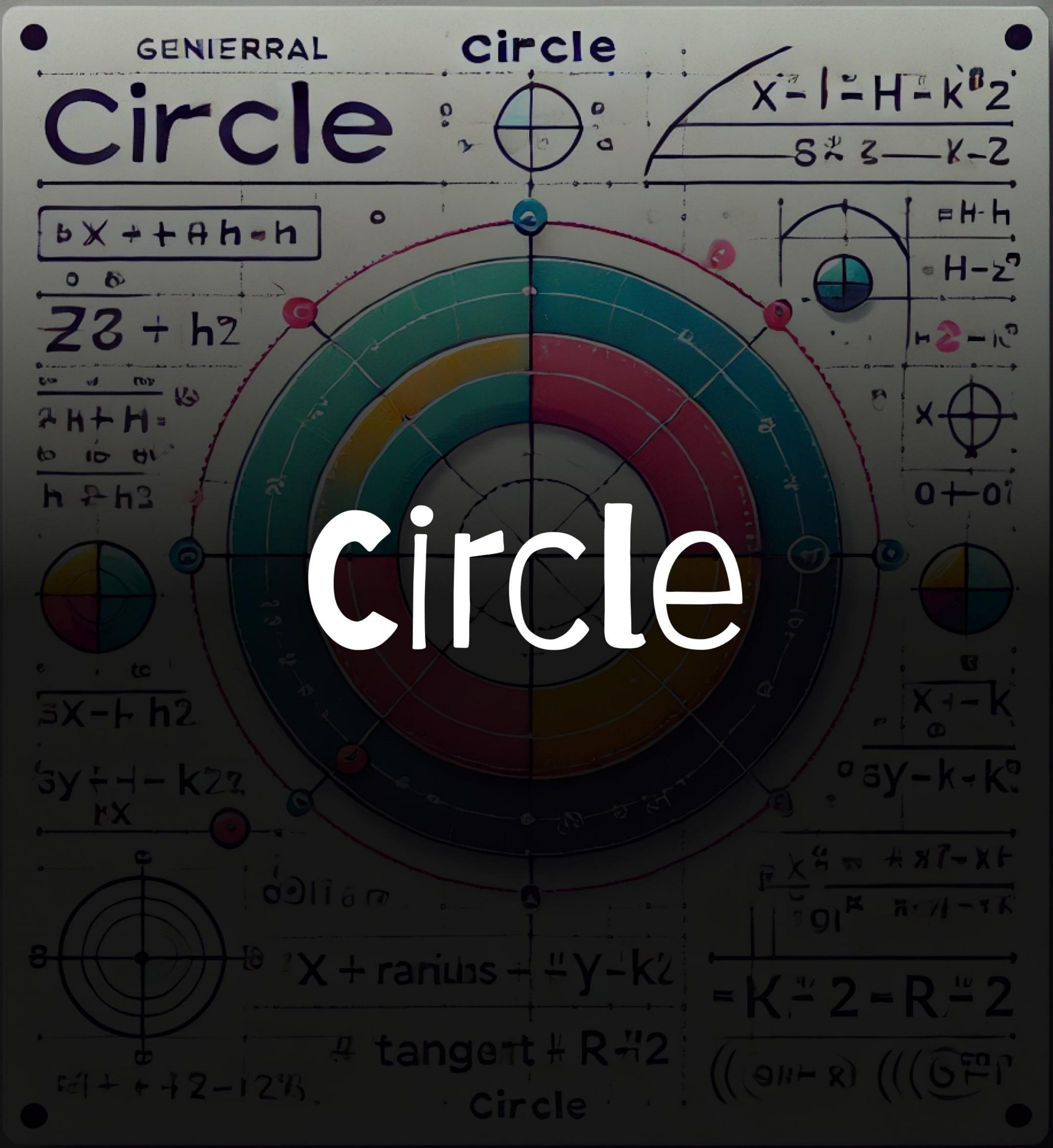


Circle

CIRCLE

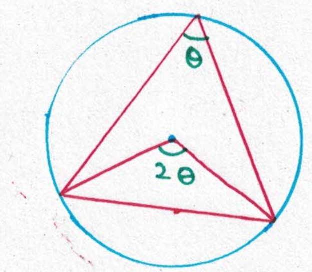
BASIC GEOMETRY

(i) Equal chords subtend equal angles at the centre

(ii) Equal chords are equidistant from center

(iii) Larger the chords, much nearer to the center

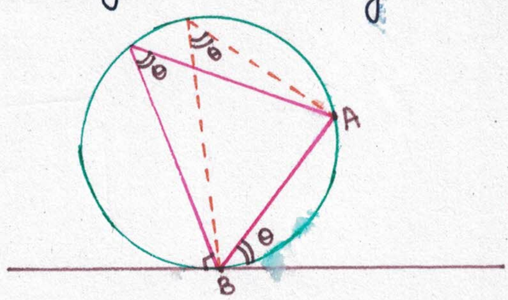
(iv) Angle subtended by the chord at the centre is double the angle made by the chord at the circumference in the same segment



(v) Angle made by chord in same segment are always equal.

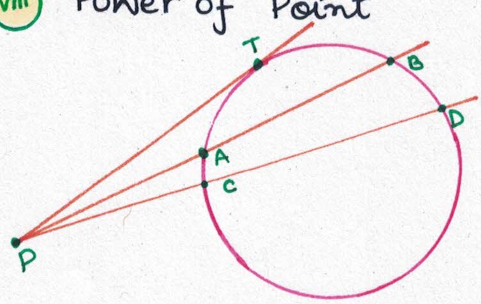
(vi) Perpendicular from center to the chord bisects the chord and vice-versa

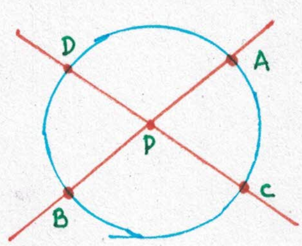
(vii) Angle made by chord with tangent at any one of its end point is equal to angle made by the chord in alternate segment.



(viii) Power of Point

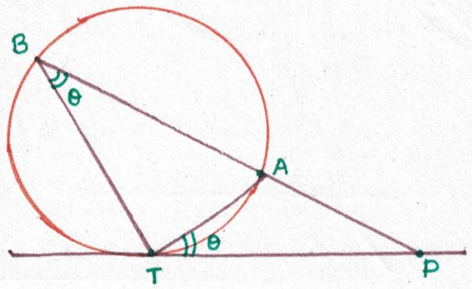
$$\boxed{PT^2 = PA \times PB = PC \times CD}$$





$PA \times PB = PC \times PD$

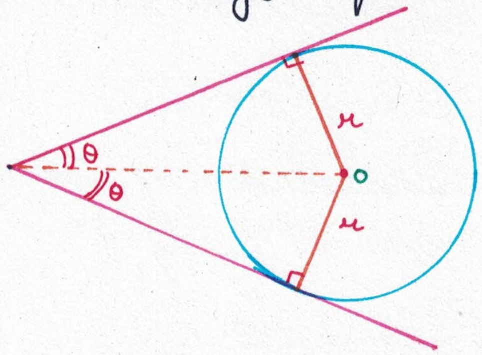
PROOF



$\Delta PTA \sim \Delta PBT$

$$\frac{PT}{PA} = \frac{PB}{PT} \Rightarrow PT^2 = PA \times PB$$

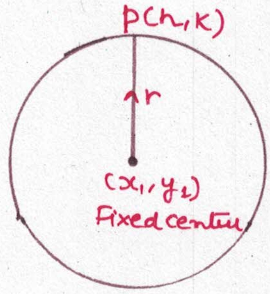
(ix) Length of tangent drawn from an external point to any circle are always equal.



$PA = PB$

DEFINATION OF CIRCLE

Locus of a point in a plane whose distance from a fixed point is always constant.



$$(x-x_1)^2 + (y-y_1)^2 = r^2 \quad \text{— ①}$$

(x_1, y_1) = centre

r = radius

General 2nd degree curve

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represent a circle if -

(i) Coeff. of x^2 = coeff. of y^2

(ii) Coeff. of $xy = 0$

General Eqn of Circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{— ②}$$

Compare 1 and 2,

Center = $(-g, -f)$

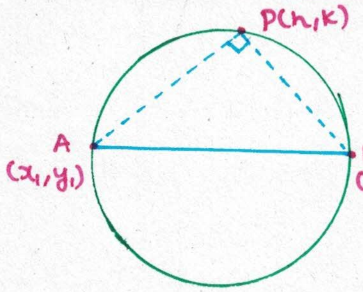
radius = $\sqrt{g^2+f^2-c}$

If $g^2+f^2-c > 0 \longrightarrow$ Real circle

If $g^2+f^2-c = 0 \longrightarrow$ Point circle

If $g^2+f^2-c < 0 \longrightarrow$ Imaginary circle with real center $(-g, -f)$

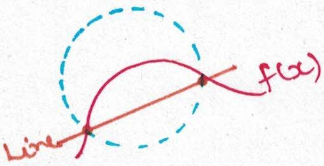
Diametric Form of a Circle



$m_1 \cdot m_2 = -1$

$$\boxed{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0}$$

$$\{\underbrace{x^2-x(x_1+x_2)+x_1x_2}_{\text{roots of } x_1, x_2}\} + \{\underbrace{y^2-y(y_1+y_2)+y_1y_2}_{\text{roots } y_1, y_2}\} = 0$$

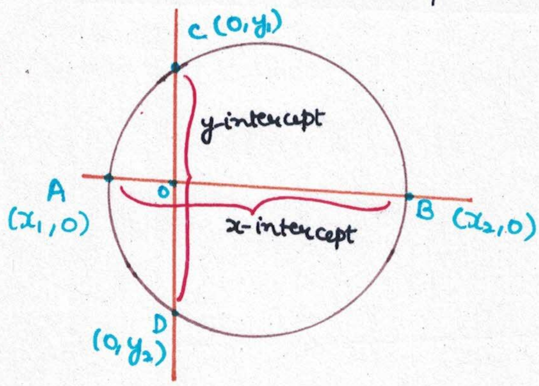


⇒ Make quad. in x, make quad. in y and add them

☞ Of all the circles passing through A and B, diametric form of circle gives the one having least area/perimeter/radius.

Intercepts

Distance between the points where circle cuts x and y axes is called x- intercept and y- intercept of circle respectively.



$x^2+y^2+2gx+2fy+c=0$

X- Intercept:

Put $y=0$,

$x^2+2gx+c=0$

$$\boxed{X \text{ int.} = |x_2-x_1| = 2\sqrt{g^2-c}}$$

Similarly,

$$\boxed{Y \text{ int.} = 2\sqrt{f^2-c}}$$

If $(g^2-c)>0$, $(f^2-c)>0$ → Real intercepts

if $(g^2-c)=0$ → circle touches x-axis

if $(f^2-c)=0$ → circle touches y-axis

if $(g^2-c)<0$ → doesn't touch x-axis

if $(f^2-c)<0$ → doesn't touch y-axis

Ques: Find equation of circle which:-

① If concentric with $3x^2+3y^2-5x+6y-14=0$ and where perimeters of semicircle is 36 units.

Sol: $3x^2+3y^2-5x+6y-14=0$

$$\downarrow$$

$$C=\left(\frac{5}{6},-\frac{6}{6}\right)=\left(\frac{5}{6},-1\right)$$

$$2r+\pi r=36$$

$$r\left(2+\frac{22}{7}\right)=36 \Rightarrow r=7$$

$$\boxed{\text{Circle: } \left(x-\frac{5}{6}\right)^2+(y+1)^2=(7)^2}$$

② Whose center is on x-axis, passes through 2,3 and radius 5

Sol: $C=h,0$

$$(x-h)^2+y^2=(5)^2$$

$$(2-h)^2+9=25$$

$$2-h=\pm 4 \Rightarrow h=2,-2 \text{ or } 6$$

$$\boxed{(x+2)^2+y^2=(5)^2 \text{ or } (x-6)^2+y^2=(5)^2}$$

③ Whose $C=(4,3)$ and touches line $5x-12y-10=0$

Sol: $$r=\left|\frac{5(4)-12(3)-10}{13}\right|=\left|\frac{-26}{13}\right|=2$$

$$\boxed{(x-4)^2+(y-3)^2=4}$$

④ Circumscribing Δ formed by lines

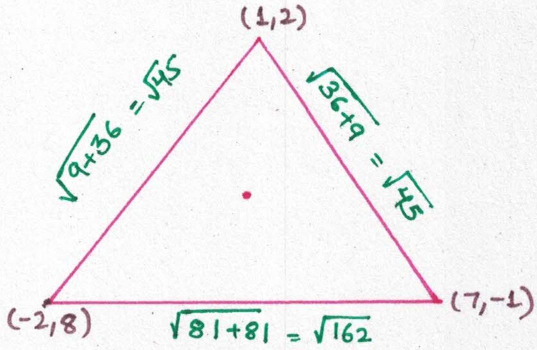
$2x+y=4$, $x+2y=5$, $x+y=6$

Sol: $12-2y+y=4 \Rightarrow y=8,\ x=-2$

$6-y+2y=5 \Rightarrow y=-1,\ x=7$

$10-3y=4 \Rightarrow y=2,\ x=1$

$(-2,8),\ (1,2),\ (7,-1)$



$x^2+y^2+2gx+2fy+c=0$

X Lengthy Method

Short Method: Any 2 degree curve passing through 3 line is:

$l_1l_2+\lambda l_2l_3+\mu l_3l_1=0$

$\Rightarrow (2x+y-4)(x+2y-5)+\lambda(x+2y-5)(x+y-6)+\mu(x+y-6)(2x+y-4)=0$

$\Rightarrow x^2(2+\lambda+2\mu)+y^2(2+2\lambda+\mu)+xy(5+3\lambda+3\mu)+\ldots=0$

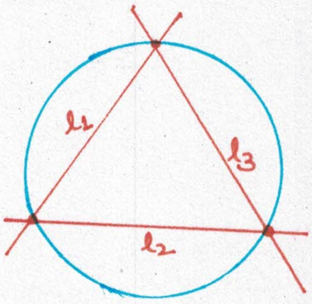
$2+\lambda+2\mu - 2+2\lambda+\mu$

$\mu=\lambda$

$2+3\left(\frac{-5}{6}\right)=-\frac{1}{2}$

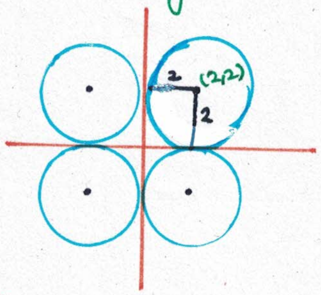
$5+6\mu=0$

$\mu=\frac{-5}{6}=\lambda$



$\Rightarrow \boxed{-\frac{1}{2}x^2-\frac{1}{2}y^2+(-4x)-5x-5y-8y=0}$

⑤ Touching both co-ordinate axis and radius = 2



Four circles:

$\boxed{(x+2)^2+(y+2)^2=0}$

⑥ Passing through extremities of diameter of circle $x^2+y^2=4$,

$C=(-1,2)$, $x^2+y^2+2x-4y-2=0$ and $r=\sqrt{7}$, $x^2+y^2-6x-8y+10=0$

$2x-4y+2=0$

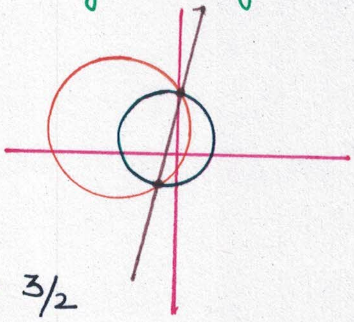
$-6x-8y+14=0$

$x-2y+1=0$

$3x+2y-7=0$

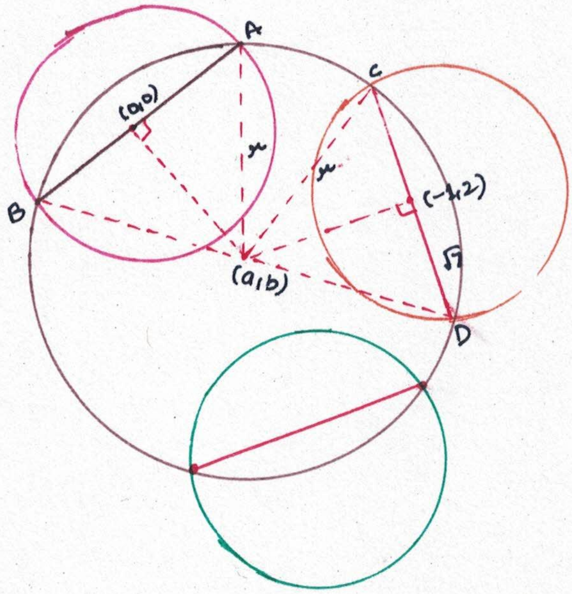
$6y-3+2y-7=0 \Rightarrow 8y=10 \Rightarrow y=5/4$, $x=3/2$

$\Rightarrow (3/2, 5/4)$



Lengthy Method

Alternate:



$a^2+b^2+4 = r^2$ — ①

$(a+1)^2 + (b-2)^2 + 7 = r^2$ — ②

$(a-3)^2 + (b-4)^2 + 15 = r^2$ — ③

On solving,

$\boxed{a = 2,\ b = 0,\ r = \sqrt{17}}$

Question: Find equation of circle which touch +ve y-axis at 4 units from origin and cuts chord of length 6 units on x-axis.

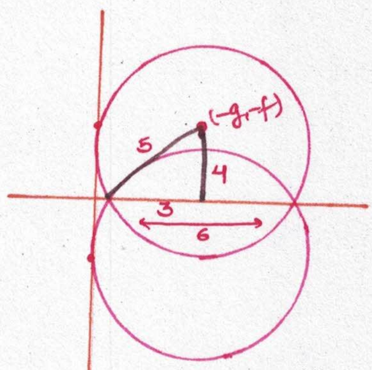
Sol: $2\sqrt{g^2-c} = 6$

$g^2 - c = 9$

$g^2+16-c = r^2$

$16+9 = r^2$

$r = 5$



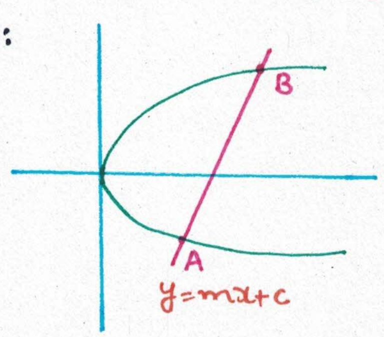
$(x-x_1)^2 + (y-4)^2 = r^2$

$(x-x_1)^2 + (y-4)^2 = 25$

$\boxed{(x\pm5)^2 + (y-4)^2 = 25}$

Ques: line $y = mx+c$ cuts the curve $y^2 = 4ax$ at A and B. Find eq^n of circle with AB as diameter.

Sol:



$(mx+c)^2 - 4ax = 0$

$m^2x^2 + x(2mc-4a) + c^2 = 0$

Two roots, x_1, x

$x = \frac{y-c}{m}$

$y^2 - 4a\left(\frac{y-c}{m}\right) = 0$

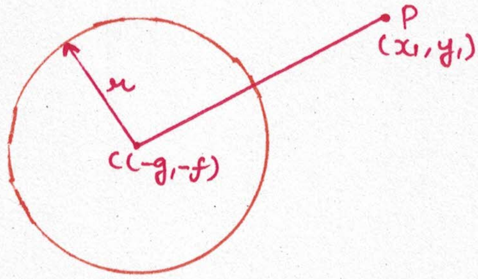
$my^2 - 4ay + 4ac = 0$

Add:

$$\boxed{x^2 + x\left(2c - \frac{4a}{m^2}\right) + y^2 - \frac{4ay}{m} + \frac{4ac}{m} = 0}$$

Position of a point with respect to a circle

$$\boxed{S = x^2 + y^2 + 2gx + 2fy + c = 0}$$



We can compare the distance of the pt. from the circle if it comes more than r, then pt. will be outside.

(i) $CP > r$

$(x_1+g)^2 + (y_1+f)^2 > g^2+f^2-c$

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c > 0$

$\boxed{S_1 > 0}$ ∴ P lies outside

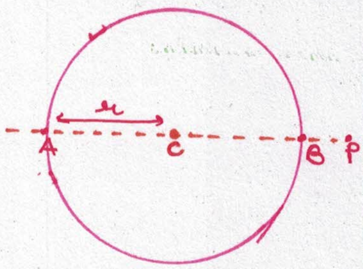
(ii) $S_1 = 0$ ∴ P lies on the circle

(iii) $S_1 < 0$ ∴ P lies inside the circle

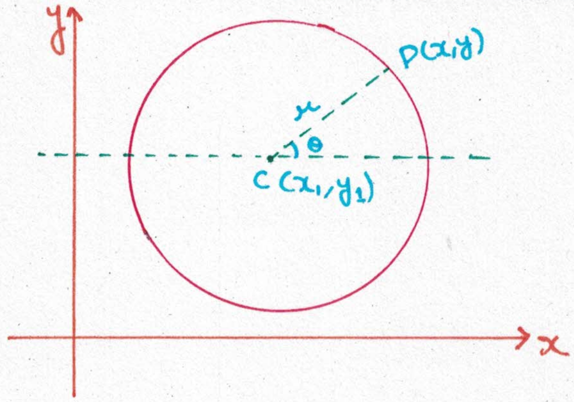
NOTE:

Maximum distance of a point from the circle = $CP + r$

Minimum distance of a point from the circle = $CP - r$



Parametric Equation of Circle



Fixed centre (x_1, y_1)

$r \rightarrow$ radius

Parameter $= \theta$

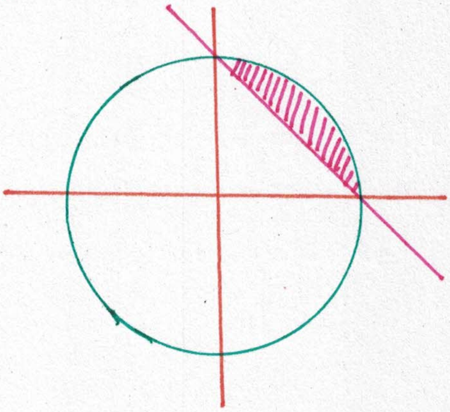
$\theta \in [0, 2\pi]$

$x = x_1 + r\cos\theta$ $\quad$ $y = y_1 + r\sin\theta$

$$\boxed{\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r} \quad \Big\}\ \text{Parametric Equation}$$

The equation looks similar to that of straight line except that here θ is parameter, there r is parameter.

Ques: Find value of a for which pt. $(a, a+1)$ lies inside the region bounded by $x^2+y^2=4$ and $x+y=2$ in Ist quadrant.



Sol:

$x+y-2>0$

$\Rightarrow \quad a+a-1-2>0$

$\Rightarrow \quad 2a-1>0$

$\Rightarrow \quad \boxed{a > \frac{1}{2}}$ — ①

$a^2+a^2+1+2a-4<0$

$\Rightarrow \quad 2a^2+2a-3<0$

$\Rightarrow \quad \dfrac{-2 \pm \sqrt{4+24}}{2\times 2} = \dfrac{-2\pm\sqrt{28}}{2\times2}$

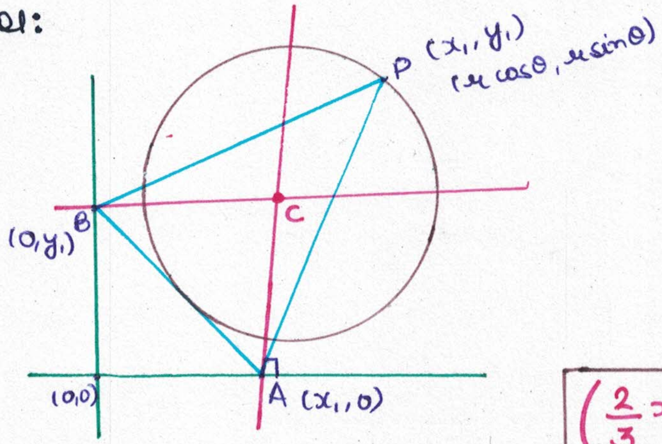
$\Rightarrow \quad \boxed{a \in \frac{-1-\sqrt{7}}{2}, \frac{-1+\sqrt{7}}{2}}$ — ②

Solve ① and ②

$$\boxed{a \in \left(\frac{1}{2}, \frac{-1+\sqrt{7}}{2}\right)}$$

Ques: 'P' is a variable pt. or circle with centre C. CA and CB are ⊥r from C on x and y axis respectively. Prove that locus of centroid of ΔAPB is circle with centre at centroid of ΔCAB and $r_1 = \frac{1}{3}$ radius of given circle.

Sol:



$G = \dfrac{2x_1 + r\cos\theta}{3}, \quad \dfrac{2y_1 + r\sin\theta}{3}$

$3h - 2x_1 = r\cos\theta$

$3K - 2y_1 = r\sin\theta$

$(2x_1 - 3h)^2 + (2y_1 - 3K)^2 = r^2$

$$\boxed{\left(\frac{2}{3}x_1 - x\right)^2 + \left(\frac{2}{3}y_1 - y\right)^2 = \left(\frac{r}{3}\right)^2}$$

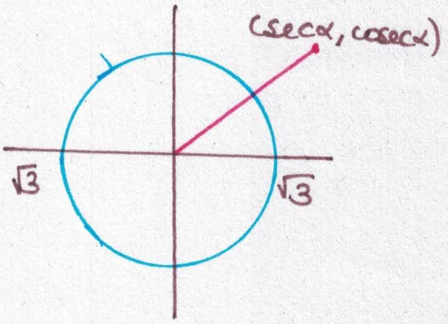
Ques: The pt. $(\sec K, \operatorname{cosec} K)$ moves in plane of circle $x^2+y^2=3$ and min. distance of this pt. from circle $= a-\sqrt{b}$ where a & b are natural no's. Find $a+b$.

$\sqrt{\sec^2\alpha + \operatorname{cosec}^2\alpha} - \sqrt{3} = \min.$

$\sqrt{\dfrac{4}{4\sin^2\alpha\cos^2\alpha}}$

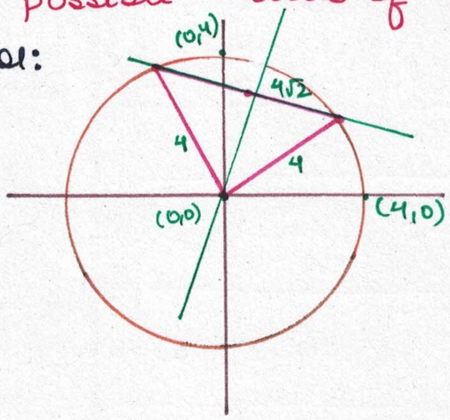
$\dfrac{2}{\sin 2\alpha} - \sqrt{3} = a - \sqrt{b}$

$a = 2,\ b = 3 \Rightarrow \boxed{a+b=5}$



Ques: Let $y = mx + c$ is chord of circle $x^2+y^2 = 16$ passing through $(1,3)$ and subtending right $\angle$ at $(0,0)$. If c_1 and c_2 are 2 possible values of c, then find (c_1+c_2)

Sol:



$3 = m + c$

$m = 3 - c$

$y = (3-c)x + c$

$\dfrac{y-(3-c)x}{c} = 1$

$\left(\dfrac{y-(3-c)x}{c}\right)^2 = x^2+y^2$

$m = -1$

$y^2 + x^2(3-c)^2 - 2xy(3-c) = cx^2 + cy^2$

$c^2 = 8(1 + (3-c)^2)$

$c_1 + c_2 = 48/7$

$\boxed{\text{Ans} \rightarrow 48}$

LINE & A CIRCLE

$C = ax + by + c^2 = 0$

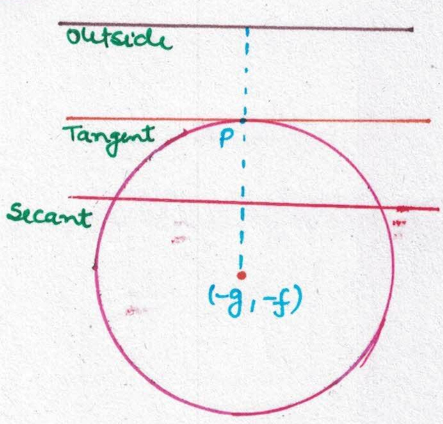
$S = x^2 + y^2 + 2gx + 2fy + c = 0$

If $p > r \rightarrow$ outside

If $p = r \rightarrow$ Tangent

condition of Tangency

If $p < r \rightarrow$ Secant



Ques: If $4l^2 - 5m^2 + 6l + 1 = 0$, then the line $lx + my + 1 = 0$ touches a fixed circle. Find its centre and radius.

Sol: $x^2+y^2+2gx+2fy+c=0$

$C = (-g, -f)$

$\Rightarrow \dfrac{-lg-mf+1}{\sqrt{l^2+m^2}} = \sqrt{g^2+f^2-c}$

$\Rightarrow (lg+mf-1)^2 = (g^2+f^2-c)(l^2+m^2)$

$\Rightarrow l^2g^2+m^2f^2+1+2lfgm-2mf-2lg = g^2l^2+g^2m^2+l^2f^2+m^2f^2-cl^2-cm^2$

$\Rightarrow m^2(g^2-c)+l^2(-c+f^2)$

$\Rightarrow 2mf = 1$

$g^2-c = -5$

$-c+f^2 = 4$

$\Rightarrow g^2+f^2 = -9$

$c=(3,0)$, $r=\sqrt{5}$

Method II:

$(lx_1+my_1+c)^2 = r^2(m^2+l^2)$

$\Rightarrow 9l^2+6l+1 = 5m^2+5l^2$

$\Rightarrow (3l+0m+1)^2 = 5(m^2+l^2)$

$\Rightarrow (x_1, y_1) = (3,0)$

$\Rightarrow r=\sqrt{5}$

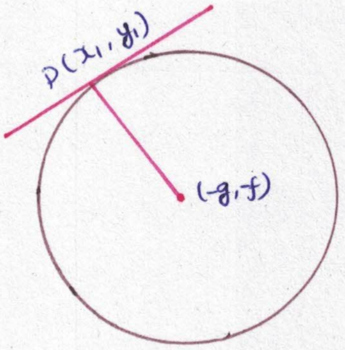
EQUATION OF TANGENTS AND NORMALS IN VARIOUS

1 Cartesian Form of Tangent

$P(x_1, y_1)$ lies on $S = x^2+y^2+2gx+2fy+c=0$

$$\boxed{xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0}$$

$T=0$



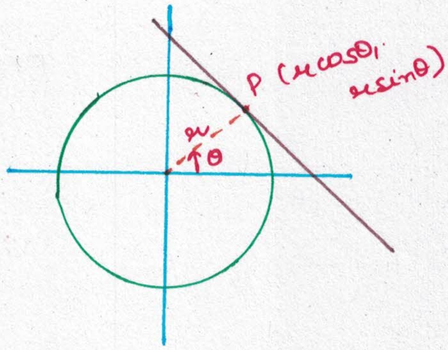
For any curve at its point (x_1, y_1) eq^n of tangent can be obtained by replacing

$x^2 \to xx_1$, $y^2 \to yy_1$

$2xy \to xy_1 + x_1y$

$2x \to x+x_1$ $\quad 2y \to y+y_1$ $\quad c \to c$

② Parametric Form of Tangent



$S = x^2 + y^2 = r^2$

$P : (r\cos\theta, r\sin\theta) \qquad \theta \in [0, 2\pi]$

Eq^n of Tangent

$$\boxed{x\cos\theta + y\sin\theta = r}$$

Just like normal form of straight line

(Where r is perpendicular distance from origin)

③ Slope Form of Tangent

$S = x^2 + y^2 = r^2$

Given slope of tangent $= m$

Let eq^n of tangent be $y = mx + c$

condition of tangency : $p = r$

$$\left|\frac{0+0+c}{\sqrt{1+m^2}}\right| = r$$

$$c = \pm r\sqrt{1+m^2}$$

Equation of tangent is :

$$\boxed{y = mx \pm r\sqrt{1+m^2}}$$

2 tangents touching at diametrical end points of circle.

if $(x-x_1)^2 + (y-y_1)^2 = r^2$

Then, the eq^n of tangent with slope m is

$$\boxed{y - y_1 = m(x - x_1) \pm r\sqrt{1+m^2}}$$

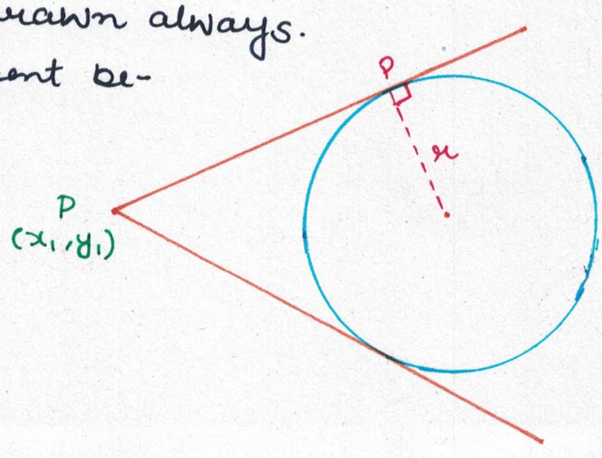
④ If 'P' lies outside the circle i.e. $S_1 > 0$

Then 2 tangents can be drawn always.

Let the equation of tangent be-

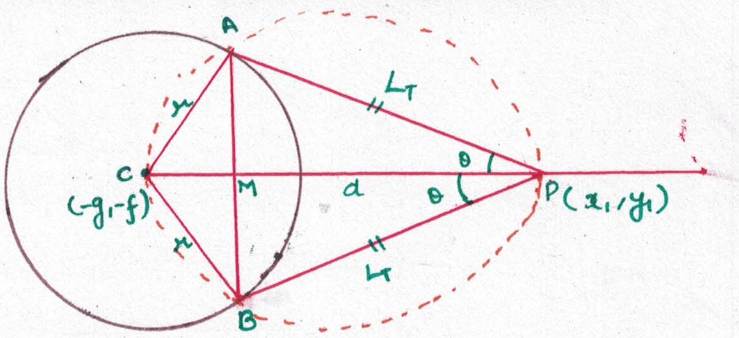
$$\boxed{y - y_1 = m(x - x_1)}$$

Apply $\boxed{p = r}$



LENGTH OF TANGENT

$$\boxed{S \rightarrow x^2+y^2+2gx+2fy+c=0}$$



(i) $L_T^2 = d^2 - r^2$

$= (x_1+g)^2 + (y_1+f)^2 - (g^2+f^2-c)$

$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$

$$\boxed{L_T = \sqrt{S_1}}$$

(ii) Area of quadrilateral PAOB $= 2\left\{\frac{1}{2} r L_T\right\}$

$= 2 \times \frac{1}{2} \times r L_T$

$$\boxed{\text{Ar } \square \text{ PAOB} = r L_T}$$

(iii) Length of chord of contact :

ΔCAP, $\tan\theta = r/L_T$

$$\boxed{AM = L_T \sin\theta}$$

$$\boxed{AB = \frac{2L_T r}{\sqrt{L_T^2+r^2}}}$$

(iv) Area of ΔPAB

(The Δ made by P and chord of contact

ar $(\Delta PAB) = \frac{1}{2} \times AB \times PM$

$\Rightarrow$ $AB = \frac{2rL_T}{\sqrt{r^2+L_T^2}}$ $\qquad PM = L_T \cos\theta$

$\Rightarrow$ ar $(\Delta PAB) = \frac{1}{2} \times 2r \frac{L_T}{\sqrt{r^2+L_T^2}} \times L_T \cos\theta$

$$\boxed{\text{ar } (\Delta PAB) = \frac{rL_T^3}{r^2+L_T^3}}$$

VI Angle between Tangents:

$$\boxed{\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}}$$ where $\tan\theta = \frac{r}{L_T}$

VI Equation of circle circumscribing ΔPAB

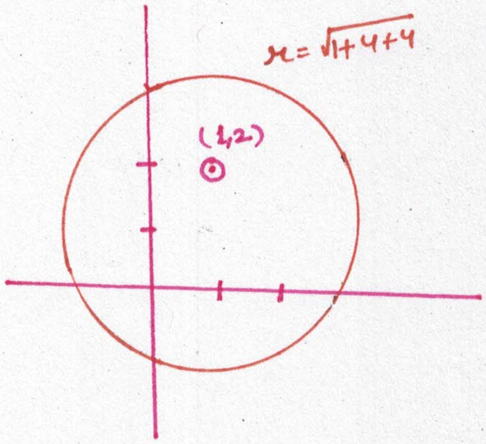
Quadrilateral PACB in cyclic CP is diameter

$$\boxed{(x-x_1)(x+g) + (y-y_1)(y+f) = 0}$$

Ques: Find area of fig. formed by the tangents to the circle - $x^2+y^2-2x-4y-4=0$, which are parallel and perpendicular to line $3x-4y-7=0$. Also find their eqⁿ.

Sol: $l_1 = 3x-4y+c' = 0$

$l_2 = -4x-3y+\lambda=0$



$$\left|\frac{3-8+c'}{5}\right| = 3$$

$c' = \pm 15+5$

$c' = 20$ or -10

$$c-c' = \frac{30}{5} = 6$$

$$\left|\frac{-4-6+\lambda}{5}\right| = 3$$

$\lambda = 10\pm 15$

$\lambda = -5$ or $\lambda = 25$

$$\frac{25+5}{5} = 6$$

$$\boxed{ar = 6\times 6 = 36}$$

Ques: Find eqⁿ of tangents drawn to circle $x^2+y^2-6x+4y-3=0$, which passes through point (7,4).

Sol: $C = (3,-2)$

$r = \sqrt{9+4+3} = 4$

$y-4 = m(x-7)$

$y = mx+4-7m$

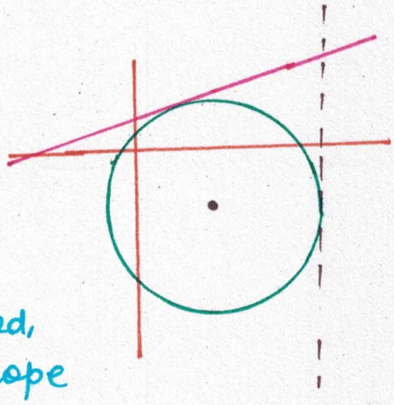
$$\frac{3m+2+4-7m}{\sqrt{1+m^2}} = \pm 4$$

$-4m+6 = \pm 4\sqrt{1+m^2}$

$-2m+3 = 2\sqrt{1+m^2}$

$4m^2+9-12m = 4+4m^2$

$5 = 12m$



★ $\boxed{m=\infty}$ $\Longrightarrow$ If m is being cancelled, then remember, slope of one line will be ∞

$y = \frac{5}{12}x + 4 - \frac{35}{12}$

$\boxed{12y = 5x-13}$

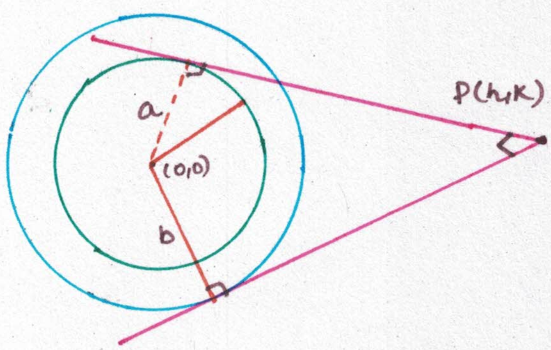
$\boxed{x=7}$

Ques: Tangent drawn to concentric circle $x^2+y^2=a^2$ and $x^2+y^2=b^2$ are at $90°$. Find locus of their point of intersection.

sol:

$y = mx+c$

$y = -\frac{x}{m}+c$



$\Rightarrow \frac{c}{\sqrt{1+m^2}} = a^2$

$\Rightarrow \frac{cm}{\sqrt{m^2+1}} = b^2$

$\Rightarrow \frac{1}{m} = \frac{a^2}{b^2}$

$\Rightarrow K = mh+c$

$\Rightarrow K = -\frac{h}{m} + K - mh$

$\Rightarrow K = -\frac{ha^2}{b^2} + K - \frac{b^2}{a^2}h$

$\Rightarrow a^2b^2K = -ha^4 + Ka^2b^2 - b^4h$

$\boxed{x^2+y^2 = a^2+b^2}$

Ques: Tangent drawn from P(4,0) to $x^2+y^2=8$ touches it at A in Ist quardrant. Find co-ordinates of B on circle such that AB = 4

sol: $L_T = \sqrt{16+16-8} = \sqrt{24}$

$y-0 = m(x-4)$

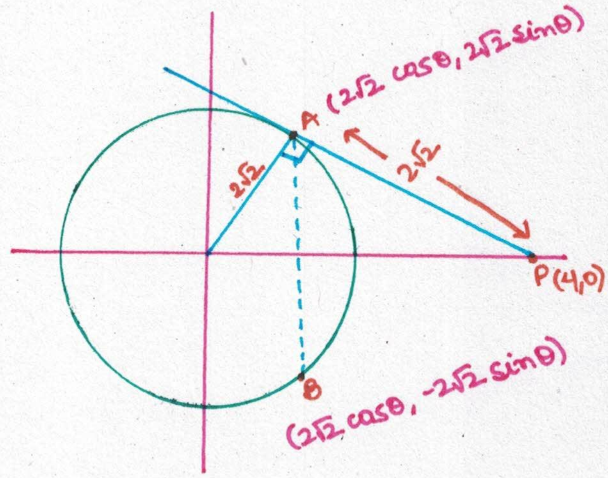
$y = mx-4m$

$4m/\sqrt{1+m^2} = 2\sqrt{2}$ $\Rightarrow 16m^2 = 8+8m^2 \Rightarrow m=-1$

$$\frac{2\sqrt{2}\sin\theta}{2\sqrt{2}\cos\theta-4} = 1$$

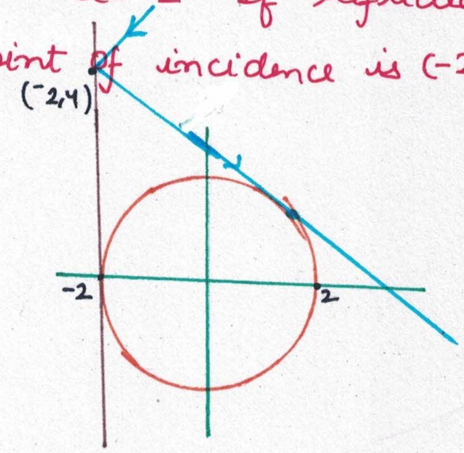
$$2\sqrt{2}\,(s-c) = -4^2$$

$$(s-c) = -\sqrt{2}$$



$$\boxed{\begin{aligned} A &= 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right), 2\sqrt{2}\left(\tfrac{1}{\sqrt{2}}\right) \\ B &= (2,-2) \\ \text{or } B &= (-2,2) \end{aligned}}$$

Ques: A light ray gets reflected from line $x=-2$. If reflected ray touches the circle $x^2+y^2=4$ and point of incidence is $(-2,4)$. Find eq^n of incident ray.



Sol:

$$y-4 = m(x+2)$$

$$y = mx+2m+4$$

$$\frac{2m+2}{\sqrt{1+m^2}} = 2$$

$$m^2+4+4m = 1+m^2$$

$$m = -\frac{3}{4}$$

$m_{\text{incident}} = 3/4$

Equation, $y-4 = \frac{3}{4}(x+2)$

$$4y = 3x+22$$

Ques: A particle moving on circle $x^2+y^2+6x-2y+8=0$ leaves its circumference at same pt. and move along its tangential direction passing through (0,0).

(i) Find equation of line pair on which particle moves

(ii) Area of quar. OACD

Sol. C = centre

A, D = pt. where particle leaves

$C = (-3,1)$

$r = \sqrt{9+1-8} = \sqrt{2}$

$$\frac{-3m-1}{\sqrt{1+m^2}} = \sqrt{2}$$

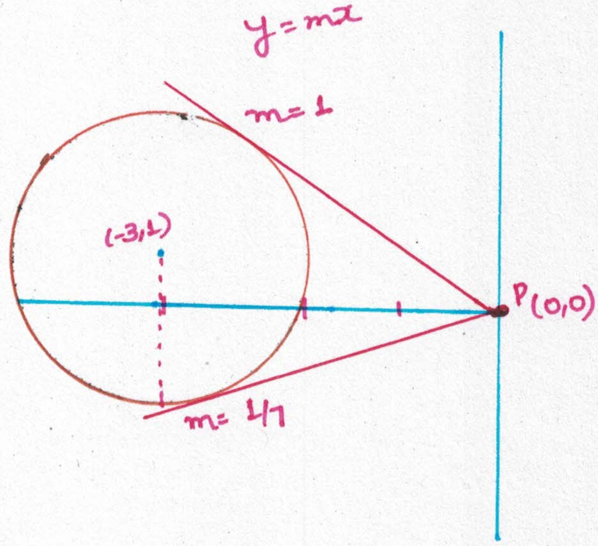
$9m^2+1+6m = 2+2m^2$

$7m^2+6m-1=0$

$7m^2+7m-m-1=0$

$(7m-1)(m+1)=0$

$m = +\frac{1}{7} \qquad m=-1$



(i) $\left(y-\frac{x}{7}\right)(y+x)=0$

$y^2-\frac{x^2}{7}+xy\left(\frac{6}{7}\right)=0$

$$\boxed{7y^2-x^2+6xy=0}$$

(ii) $y=-x$

$x^2+x^2+6x+2x+8=0$

$2x^2+8x+8=0$

$x^2+4x+4=0$

$(x+2)^2=0$

$x=-2$

$A=(-2,2)\ ,\quad y=\frac{1}{7}x$

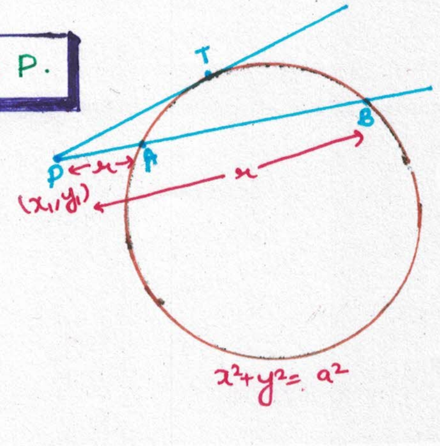
$x^2+\frac{x^2}{49}+6x-\frac{2x}{7}+8=0$

$$\boxed{\text{area} = \sqrt{2}\,L_T = \sqrt{2}\sqrt{8} = 4}$$

POWER of POINT

$$\boxed{\text{Square of length of tangent} = S_1 = \text{P.O.P.}}$$

$$\boxed{PT^2 = PA \times PB}$$



Any point on secant

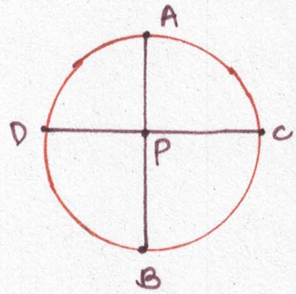
$(x_1+r\cos\theta,\ y_1+r\sin\theta)$

$(x_1+r\cos\theta)^2+(y_1+r\sin\theta)^2=a^2$

$r^2 + r \quad + (x_1^2 + y_1^2 - a^2) = 0 \rightarrow r_1, r_2$

$r_1 r_2 = x_1^2 + y_1^2 - a^2$

$\boxed{PT^2 = S_1}$



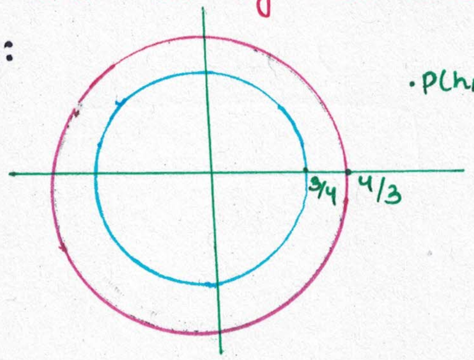
If pt. is inside,

power of pt. $= S_1$

$PA \times PB = PC \times PD$

Ques: locus of the point, from which tangent to circles, $4x^2 + 4y^2 = 9$ and $9x^2 + 9y^2 = 16$ are in ratio 3:4.

Sol:



$$\frac{\sqrt{h^2 + k^2 - \frac{9}{4}}}{\sqrt{h^2 + k^2 - \frac{16}{9}}} = \frac{3}{4}$$

$$h^2 + k^2 - \frac{9}{4} = \frac{9}{16}\left(h^2 + k^2 - \frac{16}{9}\right)$$

$$4(4h^2 + 4k^2 - 9) = 9h^2 + 9k^2 - 16$$

$$7h^2 + 7k^2 - 36 + 16 = 0$$

$$7h^2 + 7k^2 = 20$$

$$x^2 + y^2 = 20/7$$

$$\boxed{r = \sqrt{20/7}}$$

Ques: Find value of 'P' for which power of pt. P(2,5) is -ve w.r.t. $x^2 + y^2 - 8x - 12y = P$ and c neither touches nor inter-sect co-ordinate axes?

Sol: $4 + 25 - 8(2) - 12(5) + p < 0$

$p < 47$

$r < 4$

$18 + 36 - p < 16$

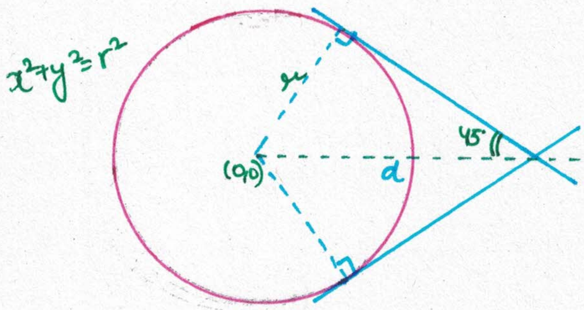
$p > 36$

$(36, 47)$

DIRECTOR CIRCLE

The locus of pt. from which tangents drawn to any curve are mutually perpendicular.

In case of circle, director circle is concentric circle whose radius is $\sqrt{2}$ times radius of given circle.



$\boxed{R_{director} = \sqrt{2}r}$

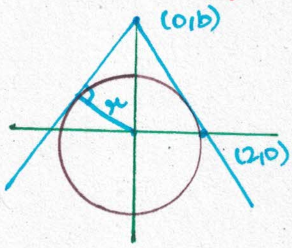
Equation of director circle:

$\boxed{x^2+y^2 = 2r^2}$

Ques: Find range of values of 'b' if angle b/w tangents drawn from (0,0) to circle $x^2+y^2=4$ is θ and θ satisfies - $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$

Sol:

$$b = \frac{r2\sqrt{2}}{\sqrt{3}-1}$$



$b \sin\frac{\pi}{2} = r \qquad b\sin\frac{\pi}{6} = r$

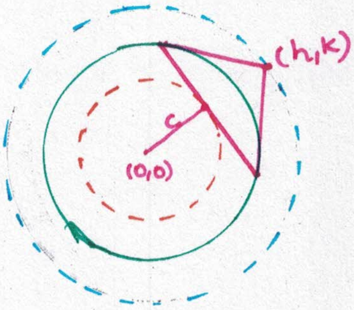
$b = 2r$

$$b \in \left(-2, \frac{-4}{\sqrt{3}}\right) \cup \left(-4, \frac{-4\sqrt{2}}{\sqrt{3}-1}\right) \cup \left(2, \frac{4}{\sqrt{3}}\right) \cup \left(4, \frac{4\sqrt{2}}{\sqrt{3}-1}\right)$$

Ques: Chord of contact drawn from point on $x^2+y^2=a^2$ to $x^2+y^2=b^2$, touches $x^2+y^2=c^2$. Find relation between a, b, c.

Sol: $h^2+k^2 = a^2$

Chord $= hx+ky-b^2$

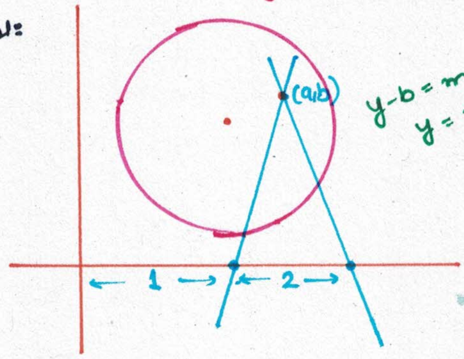


$$\frac{b^2}{\sqrt{h^2+k^2}} = c$$

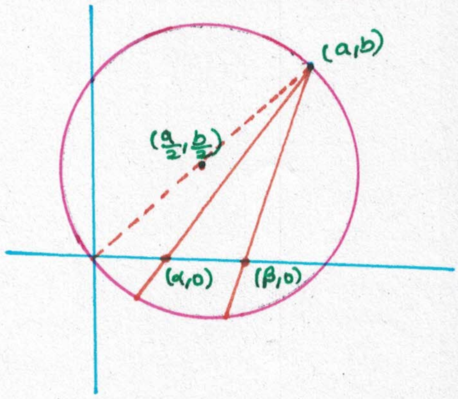
$b^2 = ca$

Ques: If 2 distinct chords of $x^2+y^2-ax-by=0$ drawn from a pt. (a,b) is divided by x-axis in ratio 2:1, Prove that, $a^2>3b^2$

Sol:



$y-b = m(x-a)$
$y = mx+b-ma$



$y = mx + b - ma$

$\Rightarrow \quad 2y + \frac{b}{3} = 0$

$\Rightarrow \quad y = \frac{-b}{2}$

$\Rightarrow \quad x^2 + \frac{b^2}{4} - ax + \frac{b^2}{2} = 0$

$\Rightarrow \quad x^2 - ax + \frac{3b^2}{4} = 0$

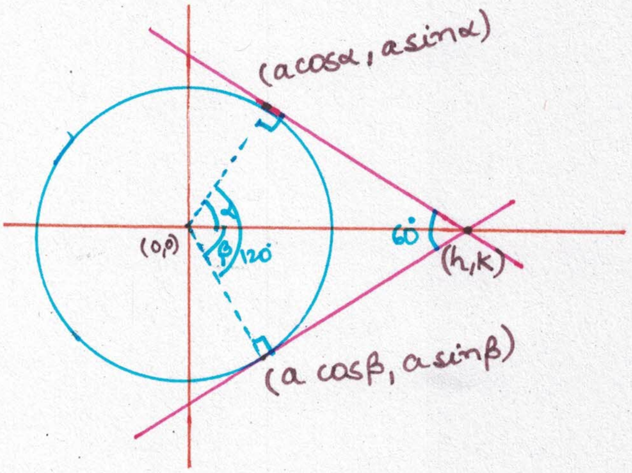
$\Rightarrow D > 0$

$\Rightarrow \quad a^2 - 3b^2 > 0$

$\Rightarrow \quad \boxed{a^2 > 3b^2}$

Ques: Find locus of point of intersection of pair of tangents drawn to circle $x^2+y^2=a^2$ at $P(\alpha)$ and $Q(\beta)$ if $|\alpha-\beta| = 120°$

Sol: $\frac{K - a\sin\alpha}{h - a\cos\alpha}$



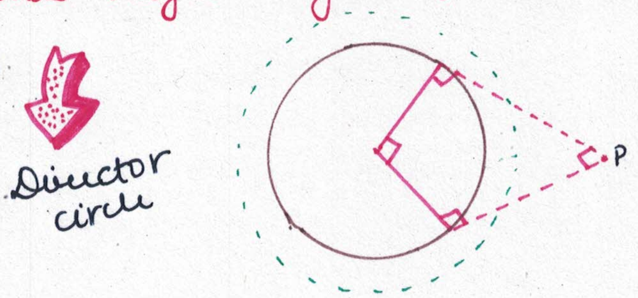
$$\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \quad \frac{a\sin\left(\frac{\alpha+\beta}{2}\right)-1}{\sin\left(\frac{\alpha-\beta}{2}\right)}$$

$$2r\cos\left(\frac{\alpha+\beta}{2}\right), \quad \frac{2}{\sqrt{3}} r \sin\left(\frac{\alpha+\beta}{2}\right)$$

$$\boxed{x^2+y^2 = 4a^2}$$

Ques: Find c.o.c of tangents drawn from P to $x^2+y^2=4$ if P subtends right angle at center. Find locus of P.

Sol:



P lies on director circle

$$\boxed{x^2+y^2 = 8}$$

PAIR OF TANGENTS

Combined equation of PA and PB

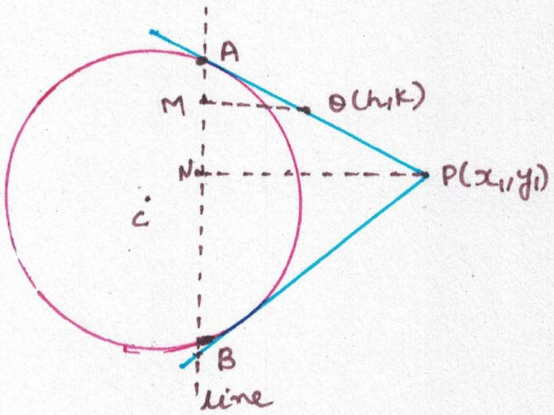
$SS_1 = T^2$

PROOF

$S = x^2+y^2-r^2$

$S_1 = x_1^2+y_1^2-r^2$

$T = xx_1+yy_1-r^2$



Chord of contact → $T=0$

$xx_1+yy_1-r^2 = 0$

$\Rightarrow \dfrac{QM}{QA} = \dfrac{PN}{PA}$

$$\Rightarrow \frac{\dfrac{hx_1+ky_1-r^2}{\sqrt{x_1^2+y_1^2}}}{\sqrt{h^2+k^2-r^2}} = \frac{\dfrac{x_1^2+y_1^2-r^2}{\sqrt{x_1^2+y_1^2}}}{\sqrt{x_1^2+y_1^2-r^2}}$$

$$\Rightarrow \underbrace{(x^2+y^2-r^2)}_{S}\underbrace{(x_1^2+y_1^2-r^2)}_{S_1} = \underbrace{(xx_1+yy_1-r^2)^2}_{T^2}$$

Ques: Tangents are drawn to the circle $x^2+y^2=a^2$ from 2 points on axis of x which are equidistant from pt. $(k,0)$. Show that locus of POI of tangent is $ky^2 = a^2(k-x)$

Sol: $T^2 = SS_1$

$(hx+ky-a^2)^2 = (x^2+y^2-a^2)(h^2+k^2-a^2)$

Put $y=0$ in this,

$\Rightarrow (hx-a^2)^2 = (x^2-a^2)(h^2+k^2-a^2)$

$\Rightarrow x^2h^2+a^4-2hxa^2 = h^2x^2+x^2(k^2-a^2)-a^2h^2-a^2(k^2-a^2)$

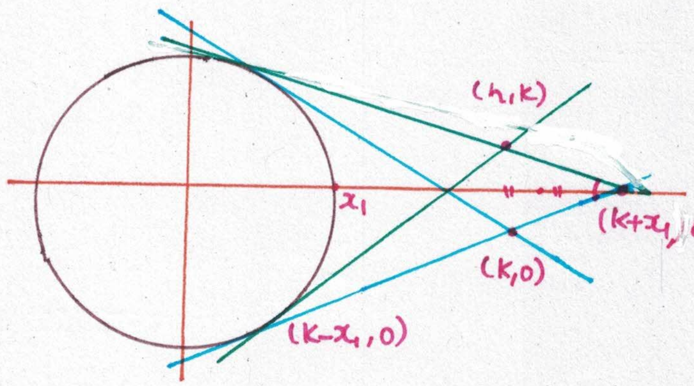
$\Rightarrow x^2(k^2-a^2)+2ha^2x-a^2(k^2-a^2+h^2)-a^4=0$

Sum of the roots:

$\Rightarrow \dfrac{-2ha^2}{y^2-a^2} = 2k$

$\Rightarrow -xa^2 = k(y^2-a^2)$

$\Rightarrow ky^2 = a^2(k-x)$

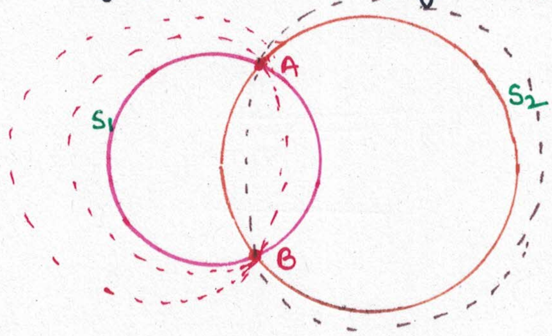


FAMILY OF CIRCLES

① Equation of all circles passing through intersection of pt. 2 given circles S_1 and S_2 can be written as-

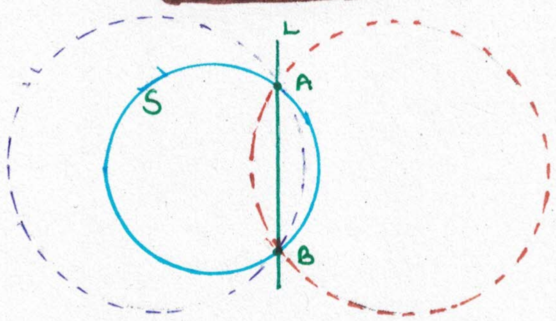
$$\boxed{S_1 + \lambda S_2 = 0} \qquad \lambda \in R$$

Where $\lambda \neq -1$ {if $\lambda = -1$ we'll get radical axis}



② Equation of all circles passing through intersection pt. of a given line and a circle can be given as-

$$\boxed{S + \lambda L = 0} \qquad \lambda \in R$$

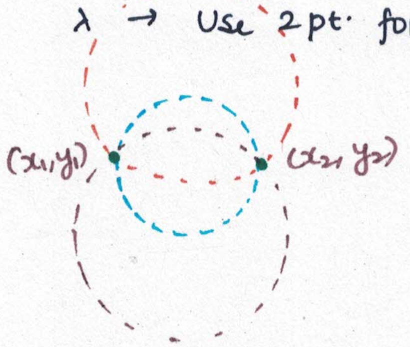


③ Equation of all circles passing through two given pts. A (x_1, y_1) and B (x_2, y_2)

$$\boxed{S + \lambda L = 0}$$

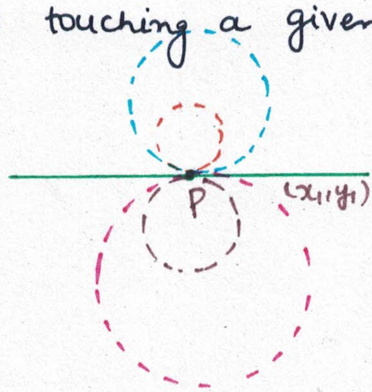
circle → diametric form

λ → Use 2pt. form



$$\boxed{(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + \lambda\left[(y-y_1) - \left(\frac{y_1-y_2}{x_1-x_2}\right)(x-x_1)\right] = 0}$$

(4) Equation of all circles touching a given line 'L' at a point 'P' on it.



S→ Assume 'P' as point circle

$$\boxed{(x-x_1)^2 + (y-y_1)^2 + \lambda(ax+by+c) = 0}$$

Ques: Find equation of circle touching a given line $2x-y-4$ at $(1,-2)$ and radius 5.

Sol: $(x-1)^2 + (y+2)^2 + \lambda(2x-y-4) = 0$

$\Rightarrow x^2+y^2 + x(-2+2\lambda) + y(-\lambda+4) - 4\lambda + 1 + 4 = 0$

$\Rightarrow (\lambda-1)^2 + \left(2-\frac{\lambda}{2}\right)^2 + 4\lambda = 30$

$\Rightarrow \lambda^2 + \frac{\lambda^2}{4} - 2\lambda + 1 + 4 - 2\lambda + 4\lambda = 30$

$\Rightarrow \frac{5\lambda^2}{4} = 25$

$\Rightarrow \boxed{x^2+y^2+x(-2+4\sqrt{5}) + y(4-2\sqrt{5}) + 5 - 8\sqrt{5} = 0}$

Ques: Show that $x^2+y^2-2x-2\lambda y-8=0$ represent family of circles passing through 2 fixed points for $\lambda \in R$. Also find eqn of the circle belonging to the family for which tangents drawn at A and B meet on line $x+2y+5=0$.

Sol: A, B - fixed pts.

S :- $(x-x_1)(x-x_2) + (y-y_1)(y-y_2)$

$-x(x_1+x_2) - y(y_1+y_2) + x_1x_2 + y_1y_2$

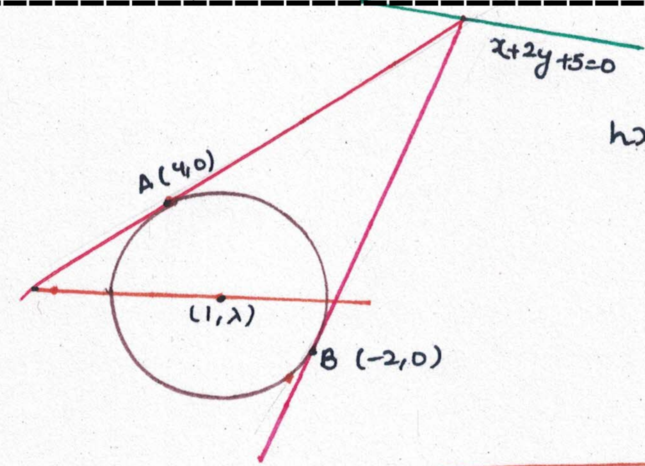
$-2y = 0$

$y = 0$

$\Rightarrow y_1 = 0, \ y_2 = 0$

$$\boxed{\begin{aligned} y_1 + y_2 &= 0 \\ y_1 &= -y_2 \end{aligned}}$$

$$\boxed{\begin{aligned} x_1x_2 + y_1y_2 &= -8 \\ x_1x_2 &= -8 \\ (2-x_1)x_1 &= -8 \\ x_1^2 + 2x_1 - 8 &= 0 \\ (x_1-4)(x_1+2) &= 0 \\ x_1 = 4, \ x_2 &= -2 \end{aligned}}$$



$hx+ky-1(x+h)-\lambda(y+k)-8=0$

$h-1=0$

$k=-3$

$-h-\lambda k-8=0$

$\lambda k=-9$

$\boxed{\lambda=3}$

$\boxed{\text{Circle- } x^2+y^2-2x-6y-8=0}$

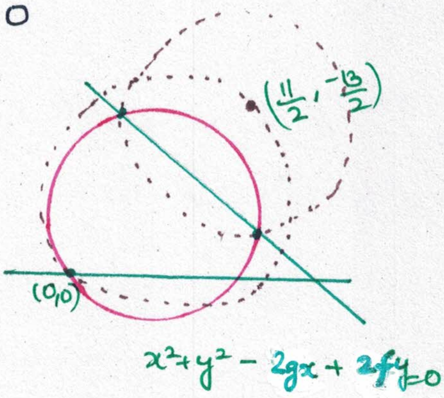
Ques: Find the eqn of the circle which pass through origin and through pt. of contanct of tangents drawn from origin to circle $x^2+y^2-11x+13y+17=0$

Sol:

$xx_1+yy_1-\frac{11}{2}(x+x_1)+\frac{13}{2}(y+y_1)+17=0$

$-\frac{11x}{2}+\frac{13y}{2}+17=0$

$\Rightarrow \quad -11x+13y+34=0$



$x^2+y^2+\lambda(-11x+13y+34)=0$

$34\lambda+17=0$

$\lambda=-1/2$

$\boxed{x^2+y^2-\frac{11x}{2}+\frac{13y}{2}=0}$

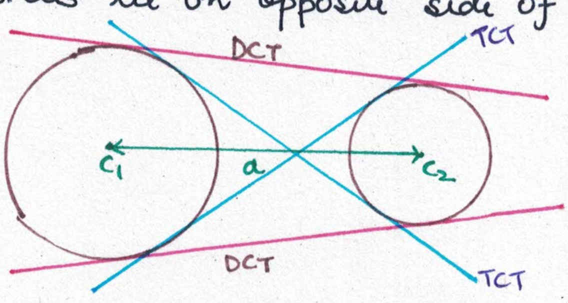
Positioning of Circle and number of Common Tangents

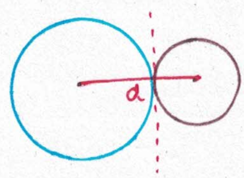
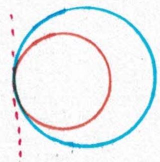
1 DCT (Direct Common Tangent

If centre of both circles lies on same side of tangent

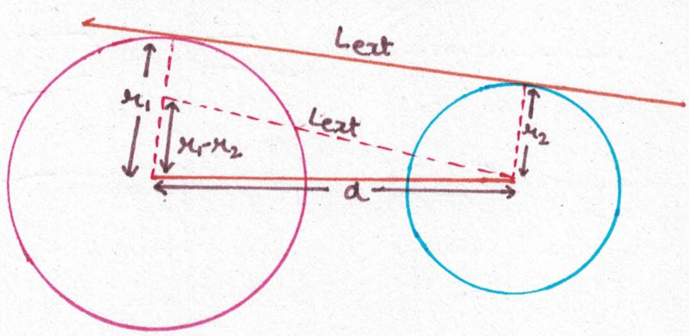
2 TCT (Transverse Common Tangent

If centre of circles lie on opposite side of tangent

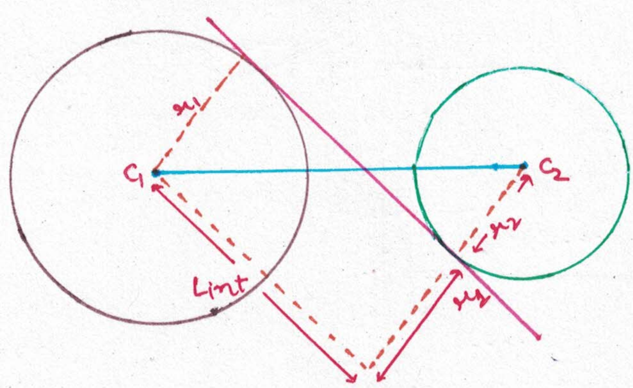


Cases	Condition	No. of Common Tangents
1 Circles are external to each other	$C_1C_2 > r_1 + r_2$ $d > r_1 + r_2$	4 → 2 DCT, 2 TCT
2 Circles are touching externally	$d = r_1 + r_2$  	3 → 2 DCT, 1 TCT
3 Circles are intersecting at 2 points	$\lvert r_1 - r_2 \rvert < d < r_1 + r_2$	2 DCT
4 Internally touching	$d = \lvert r_1 - r_2 \rvert$  	1 DCT
5 One circle is contained in another	$d < \lvert r_1 - r_2 \rvert$	No common tangent

Length Of Common Tangents



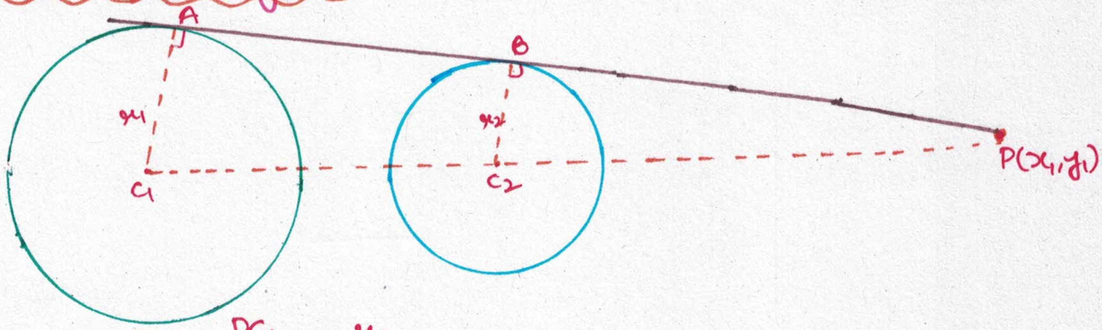
$$\boxed{L_{ext} = \sqrt{d^2 - (r_1 - r_2)^2}}$$



$$\boxed{L_{int} = \sqrt{d^2 - (r_1 + r_2)^2}}$$

Equation Of Common Tangents

For external common tangent:



$$\frac{PC_1}{PC_2} = \frac{r_1}{r_2}$$

P divides C_1C_2 in the ratio $r_1 : r_2$ externally.

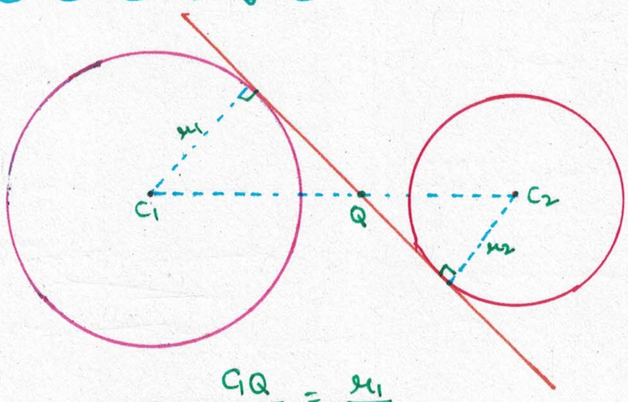
C_1, C_2 are Known, get P

Now, any line from P:-

$$y - y_1 = m(x - x_1)$$

For tangent to circle, distance of tangent from centre = radius

For internal common tangent:



$$\frac{C_1Q}{C_2Q} = \frac{r_1}{r_2}$$

Q divides C_1C_2 internally in the ratio $r_1 : r_2$

Get Q, apply condition of tangency.

P and Q are harmonic conjugate of each other with respect to centres of circles

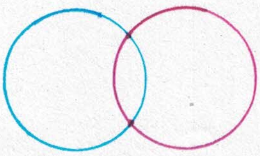
Ques: Find range of r if circles, $(x-1)^2 + (y-3)^2 = r^2$ and $(x-4)^2 + (y+1)^2 = 9$ intersect at 2pts.

$(1,3)$ $\quad$ $(4,-1)$

Sol: $|r_2 - r_1| < d < r_1 + r_2$

$|r - 3| < \sqrt{(3)^2 + (4)^2} < r + 3$

$|r - 3| < 5 < r + 3$

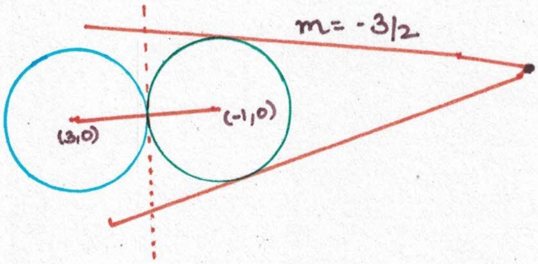


$r < 8 \qquad 2 < r$

$\Rightarrow r \in (2,8)$

Ques: Find the nature and area of Δ formed by common tangents of $x^2+y^2-6x=0$ and $x^2+y^2+2x=0$

Sol:



$d = 4$

$r_1 + r_2 = 3+1 = 4$

$r_1 = 3, \quad r_2 = 1$

$P = \dfrac{3(-1) - 1(3)}{3-1} = -3$

$y+3 = m(x)$

$\dfrac{-m-3}{\sqrt{1+m^2}} = 1$

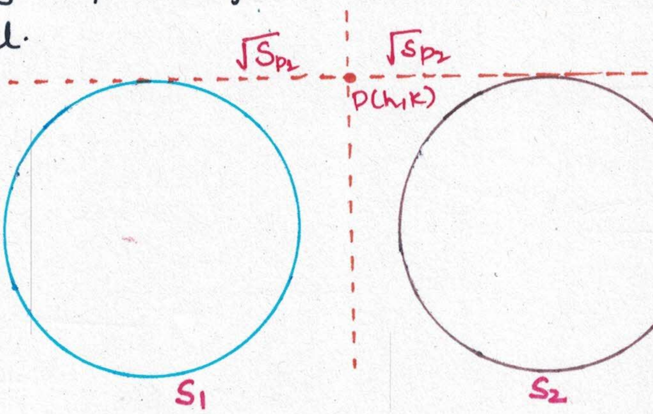
$m^2 + 9 + 6m = m^2 + 1$

$m = -3/2$

$\text{area} = \frac{1}{2} \times 2\sqrt{3} \times 3 = 3\sqrt{3}$

Radical Axis

It is the locus of a point from which power of point of 2 given circles is equal.



$S_1 = x^2+y^2+2g_1x+2f_1y+c_1 = 0$

$S_2 = x^2+y^2+2g_2x+2f_2y+c_2 = 0$

$S_1 = S_2$

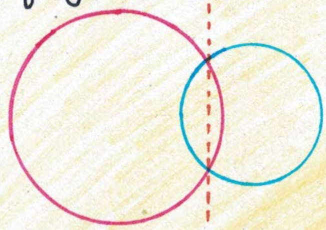
$$\boxed{2x(g_1-g_2) + 2y(f_1-f_2) + (c_1-c_2) = 0}$$

$$\underbrace{\boxed{S_1 - S_2 = 0}}_{\text{Equation of radical axis}}$$

NOTE

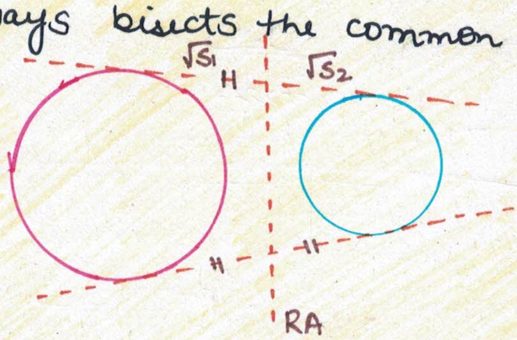
1. If 2 circles touch each other, than radical axis is common tangent at their point of contact.

2. If 2 circles intersect each other, then their common chord is radical axis of given circle.



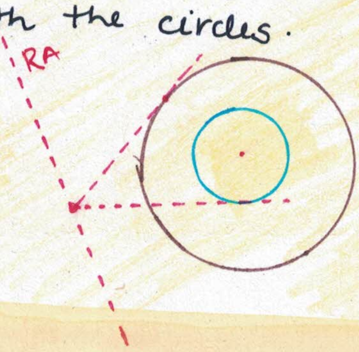
3. Radical axis is always Lr to the line joining centres of the circles

4. Radical axis always bisects the common tangent of 2 circles.



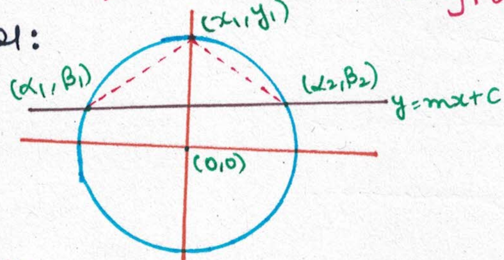
5. Radical axis is not defined only for concentric circle

6. If one circle is contained in another, then radical axis passes outside to both the circles.



Ques: Find eq. of straight line meeting circle $x^2+y^2=a^2$ at 2 points at equal distance 'd' from pt. (x_1, y_1) on circumference.

Sol:



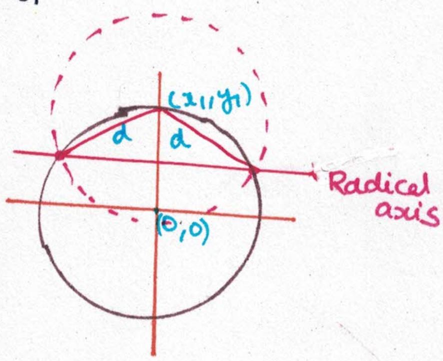
$$x^2+m^2x^2+c^2+2mcx-a^2=0$$

$$(x_1-\alpha_1)^2+(y_1-\beta_1)^2 = d$$

$$(x-x_1)^2+(y-y_1)^2 = d^2$$

$x_1^2+y_1^2=a^2$

$x_1^2+y_1^2-2xx_1-2yy_1=d^2-a^2$



Ques: Find relation between a, b, c if circles $x^2+y^2+2ax+c^2=0$ and $x^2+y^2+2bx+c^2=0$ touch each other at $(-b,0)$

Sol: $d=\sqrt{(a-b)^2+0}=(a-b)$

① $r_1+r_2=\sqrt{a^2-c^2}+\sqrt{b^2-c^2}$

② $r_1-r_2=\sqrt{a^2-c^2}-\sqrt{b^2-c^2}$

$(a-b)^2=a^2-c^2+b^2-c^2+2\sqrt{(a^2-c^2)(b^2-c^2)}$

$\Rightarrow\ -2ab=-2c^2+2\sqrt{(a^2-c^2)(b^2-c^2)}$

$\Rightarrow\ (c^2-ab)^2=(a^2-c^2)(b^2-c^2)$

$\Rightarrow\ c^4+a^2b^2-2c^2ab=a^2b^2-a^2c^2-c^2b^2+c^4$

$\Rightarrow\ a^2c^2+c^2b^2=2c^2ab$

$\Rightarrow\ \boxed{a^2+b^2=2ab}$

$(a-b)^2=0\ \Rightarrow\ a=b$

$x^2+y^2+2ax+c^2=0$ and $x^2+y^2-2by+c^2=0$

$\Downarrow$ $(-a,0)$ $\Downarrow$ $(0,-b)$

$d=\sqrt{a^2+b^2}$ $r_1+r_2=\sqrt{a^2-c^2}+\sqrt{b^2-c^2}$

$\Rightarrow\ a^2+b^2=a^2-c^2+b^2-c^2+2\sqrt{(a^2-c^2)(b^2-c^2)}$

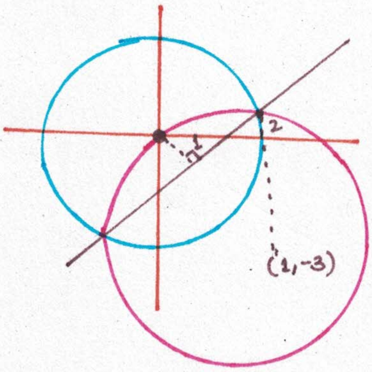
$\Rightarrow\ c^2=\sqrt{(a^2-c^2)(b^2-c^2)}$

$\Rightarrow\ c^4=a^2b^2-a^2c^2-c^2b^2+c^4$

$\Rightarrow\ a^2b^2=a^2c^2+b^2c^2$

$\Rightarrow\ \boxed{\frac{1}{c^2}=\frac{1}{b^2}+\frac{1}{a^2}}$

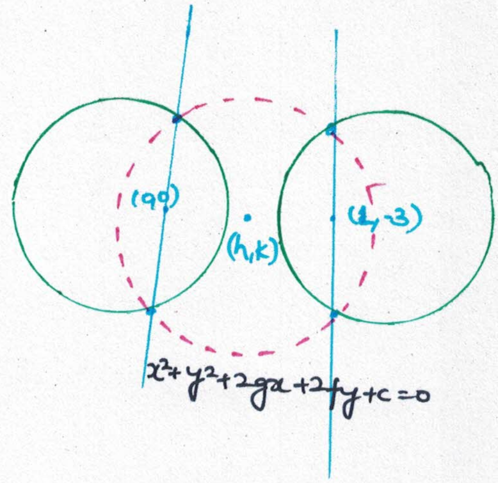
Ques: Find the locus of centre of circles which bisect the circumference of circle $x^2+y^2=4$ and $x^2+y^2-2x+6y+1=0$



$r = \sqrt{1+9-1} = 3$

$2x - 6y - 1 - 4 = 0$

$2x - 6y = 5$



$RA_2 \Rightarrow 2gx + 2fy + 2x - 6y + c - 1 = 0 \leftarrow (1,-3)$

$RA_1 \Rightarrow 2gx + 2fy + c + 4 = 0$

$\uparrow (0,0)$

$c = -4$

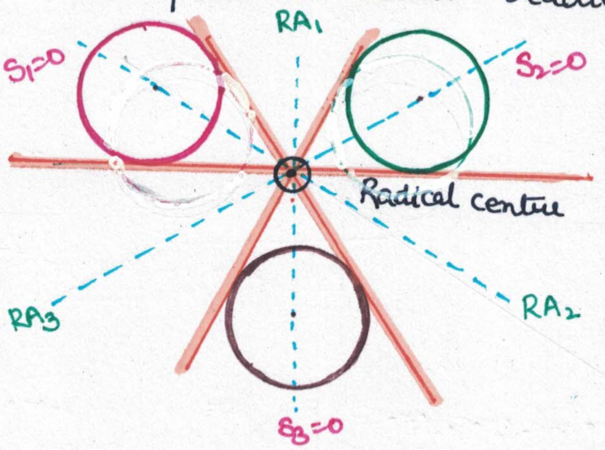
$RA_2 \rightarrow$ putting $(1,-3) \rightarrow 2g - 6f + 15x$.

$$\boxed{-2x + 6K + 15 = 0}$$

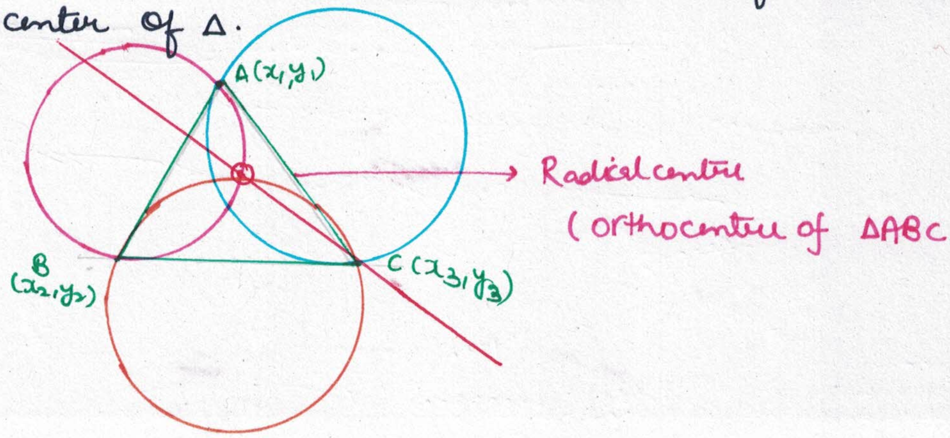
Radical Centre

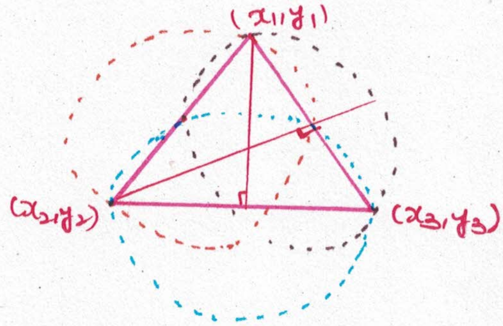
Point whose power w.r.t. three given circles is equal.

Or If tangents drawn from a pt. to 3 circles are of equal lengths, then this point is called radical centre



NOTE If Radical centre of 3 circles described on sides of ΔABC as diameter is orthocenter of Δ.





Ques: From pt. P tangents drawn to $x^2+y^2+x-3=0$, $3x^2+3y^2-5x+3y=0$, $4x^2+4y^2+8x+7y+9=0$ are of equal lengths. Find eq. of circle passing through P and touching line $x+y=5$ at $(6,-1)$

Sol:

$$C_1 :- \ x^2+y^2+x-3=0$$
$$C_2 :- \ x^2+y^2-\frac{5}{3}x+y=0$$
$$C_3 :- \ x^2+y^2+2x+\frac{7}{4}y+\frac{9}{4}=0$$

$R_{12} :- \ \frac{8x}{3}-y-3=0$

$y = \frac{8x}{3}-3$

$y=-3$

$\boxed{(0,-3)}$

$R_{C3} :- \ \frac{x}{3}+\frac{3}{4}y+\frac{9}{4}=0$

$\frac{x}{3}+\frac{3}{4}\left(\frac{8x}{3}-3\right)+\frac{9}{4}=0$

$\frac{x}{3}+2x=0$

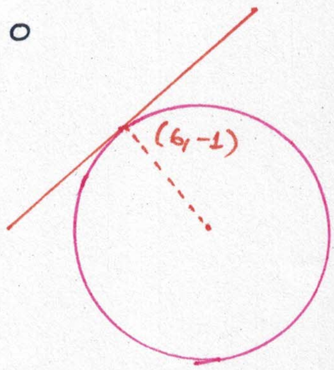
$x=0$

$(x-6)^2+(y+1)^2+\lambda(x+y-5)=0$

$36+4+\lambda(-8)=0$

$40=8\lambda \Rightarrow \lambda=5$

$$\boxed{(x-6)^2+(y+1)^2+5(x+y-5)=0}$$



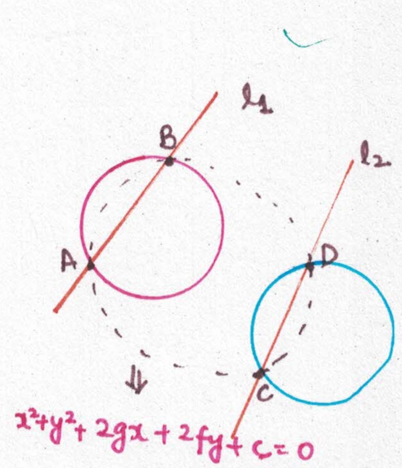
Ques: Lines $lx+my+n=0$ and $px+qy+r=0$ intersects $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ at A, B and C, D respectively. If A, B, C, D are concyclic, prove that

$$\begin{vmatrix} 2(g_1-g_2) & 2(f_1-f_2) & c_1-c_2 \\ l & m & n \\ p & q & r \end{vmatrix} = 0$$

Sol: $l_1 :- \ 2x(g-g_1)+2y(f-f_1)+c-c_1=0$

$l_2 :- \ 2x(g-g_2)+2y(f-f_2)+c-c_2=0$

$$\frac{2(g-g_1)}{l} = \frac{2(f-f_1)}{m} = \frac{c-c_1}{n} = \lambda$$

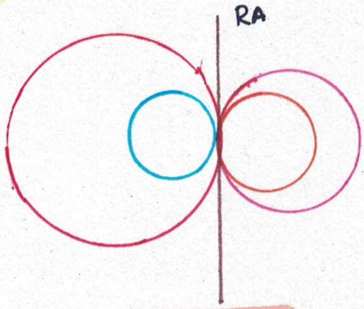


$$\frac{2(g-g_2)}{p} = \frac{2(f-f_2)}{q} = \frac{c-c_2}{r}$$

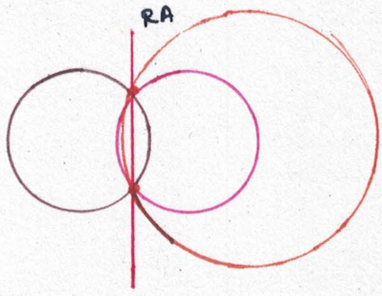
$$2(g_1-g_2)x + 2y(f_1-f_2) + c_1-c_2 = 0$$ **Proved**

Co-Axial Circles

Family of circles having same radical axis are called **Co-axial Circles.**



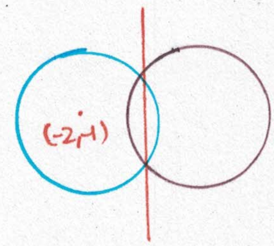
$\boxed{S+\lambda L=0}$



$\boxed{S+\lambda L=0}$

Ques: Find equation of circle co-axial with $x^2+y^2+4x+2y+1=0$ and $2x^2+2y^2-2x-4y-3=0$ and centre of that circle lies on RA. of given circles

Sol: $x^2+y^2-x-2y-\frac{3}{2}=0$



$5x+4y+\frac{5}{2}=0$

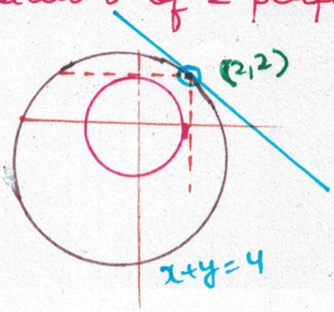
$$y = \left(\frac{\frac{5}{2}+5x}{4}\right)$$

$$= x^2+\frac{1}{16}\left(\frac{25}{4}+25x^2-10x\right)+4x-\left(\frac{\frac{5}{2}+5x}{2}\right)+120-5\left(\frac{4+5\lambda}{2}\right)-4\left(\frac{2+4\lambda}{2}\right)$$

$$+\frac{5}{2}=0$$

$$41\lambda = -23 \Rightarrow \lambda = \frac{-23}{41}$$

Ques: Find eq. of circle passing through (1,1) and belonging to co-axial circles which touches locus of point of intersection of 2 perpendicular tangents to $x^2+y^2=4$ at (2,2)



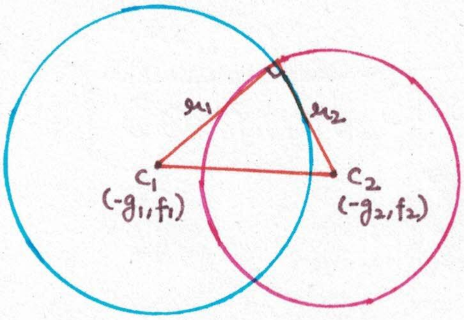
Sol: $x^2+y^2=8$

$\Rightarrow x^2+y^2-8+\lambda(x+y-4)=0$

$\Rightarrow x^2+y^2-3x-3y+4=0$

ORTHOGONALITY OF TWO CIRCLES

2 circles are orthogonal if their angle of intersection is 90°



$r_1^2 + r_2^2 = d^2$

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

If line is orthogonal to circle, then line is normal.